Denote by

- $A_+ \subset \mathbb{C}((t))$: the subalgebra $\sum_{n>0} a_n t^n$
- A_-: subalgebra $\sum_{n<0} a_n t^n$

Define

- $N_+ = [\mathfrak{g} \otimes A_+] \oplus \mathbb{C}E$,
- $N_0 = \mathbb{C}H \oplus \mathbb{C}c$,
- $N_- = [\mathfrak{g} \otimes A_-] \oplus \mathbb{C}F$

$\longrightarrow \hat{\mathfrak{g}} = N_+ \oplus N_0 \oplus N_-$

<u>Definition:</u>

Let κ and λ be complex numbers. A left $\hat{\mathfrak{g}}$-module $\hat{V}_{\kappa,\lambda}$ is "highest weight rep." with level κ and highest weight λ if:

(a) $\exists\, v \in \hat{V}_{\kappa,\lambda}$ non-zero with

$$N_+ v = 0, \quad cv = \kappa v, \quad Hv = \lambda v$$

(b) $U(N_-)$ generated by v coincides with $\hat{V}_{\kappa,\lambda}$

(a) $\Longrightarrow cu = \kappa u \quad \forall\, u \in \hat{V}_{\kappa,\lambda}$

$\hat{V}_{\kappa,\lambda}$ is generated by $a \dots a_r v$ with $a_1, \dots, a_r \in N_-$, $r \geq 0$.

Again, we have $\hat{V}_{\kappa,\lambda} = M_{\kappa,\lambda}/J$ where $M_{\kappa,\lambda}$ is Verma module.

For general values of κ and λ, $M_{\kappa,\lambda}$ is itself an irreducible $\hat{g}$ module, but for $\kappa \in \mathbb{Z}$ we have:

<u>Proposition 5:</u>

Let κ be a positive integer and λ an integer s.t. $0 \leq \lambda \leq \kappa$. Define $x \in M_{\kappa,\lambda}$ by $x = (E \otimes t^{-1})^{\kappa-\lambda+1} v$ where v is highest weight vector in $M_{\kappa,\lambda}$. Then $N_+ x = 0$ and $U(N_-)x$ is sub-module of $M_{\kappa,\lambda}$ and

$$H_{\kappa,\lambda} = M_{\kappa,\lambda}/U(N_-)x$$

is irreducible $\hat{g}$ module.

$H_{\kappa,\lambda}$ is called "integrable highest weight" module and x is called "null vector".

Dual representation:

The dual vector space H_λ^* of H_λ has the structure of a right $\hat{g}$ module:

$$\langle \xi\alpha, \eta \rangle = \langle \xi, \alpha\eta \rangle \quad \forall\, \xi \in H_\lambda^*, \eta \in H_\lambda, \alpha \in \hat{g}$$

v^* dual to $v \Rightarrow v^* N_- = 0 \Rightarrow H_\lambda^*$ is generated by v^* as right $U(N_+)$ module.

Define left-representation:

$$\rho^*(X \otimes t^n)\xi = -\xi X \otimes t^{-n}, \quad X \in g, \xi \in H_\lambda^*$$

$\longrightarrow$ dual representation of H_λ.

Action of Virasoro Lie algebra on H_λ:

Define "Sugawara operators"

$$(n \neq 0): \quad L_n = \frac{1}{2(K+2)} \sum_{j \in \mathbb{Z}} \sum_\mu I_\mu \otimes t^{-j} \cdot I_\mu \otimes t^{n+j}$$

$$(n = 0): \quad L_0 = \frac{1}{K+2} \sum_{j \geq 1} \sum_\mu I_\mu \otimes t^{-j} \cdot I_\mu \otimes t^{j}$$

$$+ \frac{1}{2(K+2)} \sum_\mu I_\mu \cdot I_\mu$$

$\longrightarrow$ defines action of Virasoro algebra on H_λ!

Proposition 6:

The Sugawara operator L_m, $m \in \mathbb{Z}$, acting on the integrable highest weight $\hat{\mathfrak{g}}$ module $H_{k,\lambda}$ satisfy

$$[L_m, X \otimes t^n] = -n X \otimes t^{m+n}, \quad X \in \mathfrak{g}$$

for any integer n. We sometimes also write this as

$$[L_m, X[n]] = -n X[m+n]$$

We also have $[L_n, c] = 0$.

Proposition 7:

As linear operators on $H_{k,\lambda}$ we have

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\frac{3k}{k+2}$$

$$\Rightarrow \quad c = \frac{3k}{k+2} \quad \text{"central charge"}$$

$H_{k,\lambda}$ contains a subspace V_λ which is finite-dim. rep. of $sl_2(\mathbb{C})$. For $u \in V_\lambda$:

$$L_0 u = \frac{1}{2(k+2)}\left(\sum_\mu I_\mu \cdot I_\mu\right) u = \frac{j(j+1)}{k+2} u$$

Have set $\lambda = 2j$. Set $\Delta \equiv \frac{j(j+1)}{k+2}$ "conformal weight"

Remark:

The above situation is for $sl_2(\mathbb{C})$.
For general Lie algebras $\mathfrak{g}$ and their affine extensions, we have:

$$L_0 = \frac{1}{2(k+h^\vee)} \sum_\mu I_\mu \cdot I_\mu v = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(k+h^\vee)}$$

dual coxeter number

where $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum \omega_i$

roots

fundamental weights

Generating functions and currents

Let $x \in \mathfrak{g}$. The current associated to x is the formal sum

$$J_x(z) = \sum_{n \in \mathbb{Z}} x[n] \cdot z^{-n-1}.$$

The current can split in two parts:

$$J_x(z) = J_x^+(z) - J_x^-(z),$$

$$J_x^+(z) = \sum_{n<0} x[n] \cdot z^{-n-1}, \quad J_x^-(z) = -\sum_{n \geq 0} x[n] \cdot z^{-n-1}$$

Let V be a highest-weight representation of $\hat{\mathfrak{g}}$ with central charge κ, and let $z \neq 0$ be a complex number. Then

$$J_x(z) : V \longrightarrow V \otimes \mathbb{C}[z, z^{-1}]$$

<u>Proposition 8:</u>

$$[J_x^\pm(z), J_y^\pm(w)] = \frac{1}{z-w}\left(J^\pm_{[x,y]}(z) - J^\pm_{[x,y]}(w)\right), \quad z \neq w$$

$$[J_x^-(z), J_y^+(w)] = \frac{1}{z-w}\left(J^-_{[x,y]}(z) - J^+_{[x,y]}(w)\right) - \frac{\kappa\langle x, y\rangle}{(z-w)^2}, \quad |z| > |w|$$

<u>Proof:</u>
straightforward calculation. $\square$

The above Prop. implies:

$$[J_x^\pm(z), J_y^\pm(z)] = \frac{d}{dz} J^\pm_{[x,y]}(z).$$

The Virasoro algebra can be obtained from the $J_a(z)$ as follows:
Define $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

Then

$$L(z) = \frac{1}{2(k+h^\vee)} \sum_\mu :J_\mu(z)^2:$$

where the "normal ordered" product $:a[n]a[k]:$ is defined by

$$:a[n]a[k]: = \begin{cases} a[n]a[k], & k \geq n, \\ a[k]a[n], & k < n. \end{cases}$$